CPCS 222
DISCRETE STRUCTURES

Dr. M. G. Abbas Malik
Assistant Professor
Faculty of Computing and IT
University of Jeddah

Picture Source: http://www.cp.eng.chula.ac.th/~atiwong/2110200/pict/header.gif
Dr. Muhammad Ghulam Abbas Malik

- PhD in Computer Science from University of Grenoble (Ex. University Joseph Fourier), France

- Areas of specialization:
  - Artificial Intelligence
  - Algorithmic Analysis
  - Machine Translation
  - Finite-state Machines
  - Machine Learning

- Contact: mgmalik@kau.edu.sa
Course: Discrete Structures

- Contact Hours (16 weeks)
  - 3 hours of Lectures per week

- Midterms
  - Sessional 1 – Weight 10%
  - Sessional 2 – Weight 15%

- Final Exam
  - Total weight 40%

- Quiz and in class assessment
  - Total weight 25%

- Lab
  - Total weight 10%

75% attendance in lectures and labs is mandatory to sit in the final exams
Course: Discrete Structures

- Text book

- Reference books
  - Discrete Mathematics, 6th Ed., Richard Johnsonbaugh
  - Discrete Mathematics, Thomas Koshy, Elsevier

- Course website
  www.sanlp.org/malik/cpcs222/2015s.html
Logic

- Cambridge Dictionary Definition: *a particular way of thinking, especially one which is reasonable and based on good judgment*

- Logic (Greek origin λογική) is the systematic study of the principles of valid inference and correct reasoning [Wikipedia]

- Logic rules give precise meaning of mathematical statements

- These rules are used to distinguish between valid and invalid mathematical arguments
A declarative sentence that is either true or false, but not both.
- Riyadh is the capital of Saudi Arabia (true)
- Adnan is the student of KAU
- All boys in Jeddah are students (false)
- $2 + 2 = 4$ (true)

A proposition is denoted by small letter $p, q, r, s$

$p, q, r, s$ are called **Boolean Variables**

**Truth value**

Truthfulness of falsity of a proposition called its truth value, denoted by T or 1 (true) and F or 0 (false) respectively.
New propositions can be developed/produced from the existing propositions and are called Compound Propositions.
Let \( p \) be a proposition. The statement 
“It is not the case that \( p \)”
is another proposition, called \textit{negation of \( p \)} and is denoted by \( \neg p \).

Example:
\( p = “\text{Today is Friday}” \)
The negation of \( p \) is:
\( \neg p = “\text{It is not the case that today is Friday}” \)
\( \neg p = “\text{Today is not Friday}” \)

\begin{tabular}{|c|c|}
\hline
\( p \) & \( \neg p \) \\
T (1) & F (0) \\
F (0) & T (1) \\
\hline
\end{tabular}

\textbf{Truth Table}

\textbf{Truth table displays relationships between the truth values of propositions}
Negation — Compound Proposition

- Negation of a proposition can also be considered the result of the negation operation.
- Negation operator (¬) constructs a new proposition from a single existing proposition and truth values of new proposition are opposite to the truth values of the existing proposition.
- Negation is unary operator.
- Logical operators are also called Connectives.
- Thus logical operators / connectives are used to build compound propositions.
AND ($\land$)

Let $p$ and $q$ are two propositions. The proposition “$p$ and $q$” denoted by $p \land q$, is the proposition that is true when both $p$ and $q$ are true and is false otherwise.

Example 1

- Cricket World 1992: Pakistan lost his four matches out of first five matches and won last two matches. Now he has to play a match with New Zealand (A team who has already won his first 7 matches). The chances of Pakistan’s qualification for Semi-Final depended on two conditions.
  1. Pakistan win from New Zealand
  2. Australia win from West Indies.
Example 1 (continued…)

1. $p = $ Pakistan win from New Zealand
2. $q = $ Australia win from West Indies

- Pakistan will qualify for Semi-Final only if these two proposition hold the **TRUE** truth value. Otherwise, Pakistan is out of the tournament.

$p \land q$ is also called conjunction of $p$ and $q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
<tr>
<td>T (1)</td>
<td>F (0)</td>
<td>F (0)</td>
</tr>
<tr>
<td>F (0)</td>
<td>T (1)</td>
<td>F (0)</td>
</tr>
<tr>
<td>F (0)</td>
<td>F (0)</td>
<td>F (0)</td>
</tr>
</tbody>
</table>
Example 2

- Intersection of two sets.
- Consider we have two set A and B, then A intersection B, denoted by $A \cap B$, is set that contains all those elements that are member of set A as well as of set B.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

1. $p$ = Element is member of A
2. $q$ = Element is member of B

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1</td>
<td>T</td>
<td>T (1)</td>
</tr>
<tr>
<td>T</td>
<td>1</td>
<td>F</td>
<td>F (0)</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>T</td>
<td>F (0)</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>F</td>
<td>F (0)</td>
</tr>
</tbody>
</table>
Let $p$ and $q$ be propositions. Proposition “$p$ or $q$”, denoted $p \lor q$, is the proposition that is false when $p$ and $q$ both are false and true otherwise.

Example 1

- I want to fuel up my car.
  1. $p =$ I have the money in my pocket.
  2. $q =$ Owner is my friend

$p \lor q$ is also called disjunction of $p$ and $q$. 

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Example 2

- Intersection of two sets.
- Consider we have two set A and B, then $A \cup B$, denoted by $A \cup B$, is set that contains all those elements that are member of set A as well as of set B.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

1. $p = \text{Element is member of A}$
2. $q = \text{Element is member of B}$
Let $p$ and $q$ be propositions. The proposition Exclusive OR of $p$ and $q$, denoted by $p \oplus q$, is the proposition that is true when exactly one of $p$ and $q$ is true and is false otherwise.

Example 1

- A printer is meant to serve $X$ and $Y$ such that it will only print a document, when only one of $X$ and $Y$ commands it. It will not print, if both $X$ and $Y$ command it at the same time.
  1. $p = X$ commands to print
  2. $q = Y$ commands to print

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \oplus q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (1)</td>
<td>T (1)</td>
<td>F (0)</td>
</tr>
<tr>
<td>T (1)</td>
<td>F (0)</td>
<td>T (1)</td>
</tr>
<tr>
<td>F (0)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
<tr>
<td>F (0)</td>
<td>F (0)</td>
<td>F (0)</td>
</tr>
</tbody>
</table>

Example 2

- One chair and two person to sit.
Let $p$ and $q$ be propositions. The implication, denoted by $p \rightarrow q$, is the proposition that is false when $p$ is true and $q$ is false, and true otherwise.

Example 1

- If Ali worked hard, then he will get an A.
  1. $p = \text{Ali worked hard}$
  2. $q = \text{Ali get an A}$

Example 2

- Ali’s dinner in a wedding
  1. $p = \text{Ali go to wedding}$
  2. $q = \text{Ali eat his dinner}$
Example 3

- If X is elected as president, then the taxes will be lowered.
  1. $p = X$ is elected as president
  2. $q = $ Taxes will be lowered

- In the implication
  - $p$ is called hypothesis, and
  - $q$ is called conclusion

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
<tr>
<td>T (1)</td>
<td>F (0)</td>
<td>F (0)</td>
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<tr>
<td>F (0)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
<tr>
<td>F (0)</td>
<td>F (0)</td>
<td>T (1)</td>
</tr>
</tbody>
</table>
IMPLICATION (→)

- There are other implications that can be formed from \( p \to q \)

**Converse**
- \( q \to p \) is called converse of \( p \to q \)

**Contrapositive**
- \( \neg q \to \neg p \) is called Contrapositive of \( p \to q \)

**Inverse**
- \( \neg p \to \neg q \) is called inverse of \( p \to q \)
**Implication (→)**

Example

“The home team wins whenever it is raining”

- **Implication**: $p \rightarrow q$
  
  if it is raining, then the home team wins.

- **Converse**: $q \rightarrow p$
  
  If the home team wins, then it is raining.

- **Contrapositive**: $\neg q \rightarrow \neg p$
  
  If the home team does not win, then it is not raining.

- **Inverse**: $\neg p \rightarrow \neg q$
  
  If it is not raining, then the home team does not win.
If $p$ and $q$ be propositions. The bidirectional of $p$ and $q$, denoted by $p \leftrightarrow q$, is true when $p$ and $q$ have the same truth values, and is false otherwise.

“$p$ if and only if $q$”

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$q \rightarrow p$</th>
<th>$p \leftrightarrow q = (p \rightarrow q) \land (q \rightarrow p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
<tr>
<td>T (1)</td>
<td>F (0)</td>
<td>F (0)</td>
<td>T (1)</td>
<td>F (0)</td>
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<tr>
<td>F (0)</td>
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<td>F (0)</td>
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</tr>
<tr>
<td>F (0)</td>
<td>F (0)</td>
<td>T (1)</td>
<td>T (1)</td>
<td>T (1)</td>
</tr>
</tbody>
</table>
# Precedence of Logical Operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Precedence</th>
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<tbody>
<tr>
<td>\neg</td>
<td>1</td>
</tr>
<tr>
<td>\land</td>
<td>2</td>
</tr>
<tr>
<td>\lor</td>
<td>3</td>
</tr>
<tr>
<td>\implies</td>
<td>4</td>
</tr>
<tr>
<td>\iff</td>
<td>5</td>
</tr>
</tbody>
</table>
Human language is often ambiguous

We can remove the ambiguity by translating the natural language (human language) into logic

Example 1:

“You can access the Internet from the campus only if you are a computer science major or you are not a fresh man”
Human language is often ambiguous

We can remove the ambiguity by translating the natural language (human language) into logic

Example 1:

“You can access the Internet from the campus only if you are a computer science major or you are not a fresh man”

p = Student can access the Internet from campus
Human language is often ambiguous
We can remove the ambiguity by translating the natural language (human language) into logic

Example 1:
“You can access the Internet from the campus only if you are a computer science major or you are not a fresh man”

\[ p = \text{Student can access the Internet from campus} \]
\[ q = \text{student is a computer science major} \]
Human language is often ambiguous
We can remove the ambiguity by translating the natural language (human language) into logic

Example 1:

“You can access the Internet from the campus only if you are a computer science major or you are not a fresh man”

- $p =$ Student can access the Internet from campus
- $q =$ student is a computer science major
- $r =$ student is in first semester
Example 1:

“You can access the Internet from the campus only if you are a computer science major or you are not a fresh man”

- p = Student can access the Internet from campus
- q = student is a computer science major
- r = student is in first semester

\[
p \rightarrow (q \lor \neg r)
\]
Example 2:
“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years”
Example 2:

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years”

▪ p = you can ride roller coaster
Example 2:

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years”

- $p = \text{you can ride roller coaster}$
- $q = \text{you are under 4 feet tall}$
Example 2:

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years”

- $p = \text{you can ride roller coaster}$
- $q = \text{you are under 4 feel tall}$
- $r = \text{you are older than 16 years}$

\[(p \land \neg q) \rightarrow \neg r\]

\[(p \land q) \rightarrow r\]
Translating sentences in natural languages into logical expressions is an essential part of specifying both hardware and software systems.

Examples:

- Boolean Searches
  1. File search
  2. Web search
- Logic puzzles

Logic and Bit Operations
An important type of step used in a mathematical arguments is the replacement of a statement with another statement with the same truth value.

**Tautology**
- A compound proposition that is always true.
  - $p \lor \neg p$

**Contradiction**
- A compound statement that is always false.
  - $p \land \neg p$
# Propositional Equivalences

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## Table 1: Examples of a Tautology and a Contradiction

<table>
<thead>
<tr>
<th></th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
<th>( p \land \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>
PROPOSITIONAL EQUIVALENCES

- Compound statements that have the same truth values in all possible cases are called logically equivalent.

- The propositions $p$ and $q$ are called logically equivalent if $p \leftrightarrow q$ is a tautology. It is denoted by $p \equiv q$.

- Sometimes the symbol $\iff$ is used instead of $\equiv$. 

Dr. M. G. Abbas Malik – FCIT, UoJ
De Morgan’s law: 
\[ \neg(p \land q) \equiv (\neg p \lor \neg q) \]
\[ \neg(p \lor q) \equiv (\neg p \land \neg q) \]

TABLE 3  Truth Tables for \( \neg(p \lor q) \) and \( \neg p \land \neg q \).

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>q</td>
<td>p \lor q</td>
<td>\neg(p \lor q)</td>
<td>\neg p</td>
<td>\neg q</td>
<td>\neg p \land \neg q</td>
</tr>
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<td>T</td>
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<td>T</td>
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</tr>
</tbody>
</table>
## Propositional Equivalences

- \( \neg p \lor q \equiv p \rightarrow q \)

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### Table 4: Truth Tables for \( p \rightarrow q \lor q \) and \( p \rightarrow q \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg p \lor q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<td>F</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Distributive Law: \[ p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \]

**TABLE 5** A Demonstration That \( p \lor (q \land r) \) and \( (p \lor q) \land (p \lor r) \) Are Logically Equivalent.
# PROPOSITIONAL EQUIVALENCES

## TABLE 6 Logical Equivalences.

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>p ∧ T = p</code></td>
<td>Identity laws</td>
</tr>
<tr>
<td><code>p ∨ F = p</code></td>
<td></td>
</tr>
<tr>
<td><code>p ∨ T = T</code></td>
<td>Domination laws</td>
</tr>
<tr>
<td><code>p ∧ F ≡ F</code></td>
<td></td>
</tr>
<tr>
<td><code>p ∨ p ≡ p</code></td>
<td>Idempotent laws</td>
</tr>
<tr>
<td><code>p ∧ p = p</code></td>
<td></td>
</tr>
<tr>
<td><code>¬(¬p) = p</code></td>
<td>Double negation law</td>
</tr>
<tr>
<td><code>p ∨ q = q ∨ p</code></td>
<td>Commutative laws</td>
</tr>
<tr>
<td><code>p ∧ q = q ∧ p</code></td>
<td></td>
</tr>
<tr>
<td><code>(p ∨ q) ∨ r = p ∨ (q ∨ r)</code></td>
<td>Associative laws</td>
</tr>
<tr>
<td><code>(p ∧ q) ∧ r ≡ p ∧ (q ∧ r)</code></td>
<td></td>
</tr>
<tr>
<td><code>p ∨ (q ∧ r) ≡ (p ∨ q) ∧ (p ∨ r)</code></td>
<td>Distributive laws</td>
</tr>
<tr>
<td><code>p ∧ (q ∨ r) = (p ∧ q) ∨ (p ∧ r)</code></td>
<td>De Morgan's laws</td>
</tr>
<tr>
<td><code>¬(p ∧ q) = ¬p ∨ ¬q</code></td>
<td></td>
</tr>
<tr>
<td><code>¬(p ∨ q) ≡ ¬p ∧ ¬q</code></td>
<td></td>
</tr>
<tr>
<td><code>p ∨ (p ∧ q) ≡ p</code></td>
<td>Absorption laws</td>
</tr>
<tr>
<td><code>p ∧ (p ∨ q) = p</code></td>
<td></td>
</tr>
<tr>
<td><code>p ∨ ¬p = T</code></td>
<td>Negation laws</td>
</tr>
<tr>
<td><code>p ∧ ¬p = F</code></td>
<td></td>
</tr>
</tbody>
</table>

## TABLE 7 Logical Equivalences Involving Conditional Statements.

<table>
<thead>
<tr>
<th>Conditional Statements</th>
<th>Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>p → q</code></td>
<td><code>¬p ∨ q</code></td>
</tr>
<tr>
<td><code>p → q</code></td>
<td><code>¬q → ¬p</code></td>
</tr>
<tr>
<td><code>p ∨ q</code></td>
<td><code>≡ p → q</code></td>
</tr>
<tr>
<td><code>p ∧ q</code></td>
<td><code>≡ (p → ¬q)</code></td>
</tr>
<tr>
<td><code>¬(p → q)</code></td>
<td><code>≡ p ∧ ¬q</code></td>
</tr>
<tr>
<td><code>(p → q) ∧ (p → r)</code></td>
<td><code>≡ p → (q ∧ r)</code></td>
</tr>
<tr>
<td><code>(p → r) ∧ (q → r)</code></td>
<td><code>≡ (p ∨ q) → r</code></td>
</tr>
<tr>
<td><code>(p → q) ∨ (p → r)</code></td>
<td><code>≡ p → (q ∨ r)</code></td>
</tr>
<tr>
<td><code>(p → r) ∨ (q → r)</code></td>
<td><code>≡ (p ∧ q) → r</code></td>
</tr>
</tbody>
</table>
# Propositional Equivalences

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## Table 8 Logical Equivalences Involving Biconditionals.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) )</td>
<td></td>
</tr>
<tr>
<td>( p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q )</td>
<td></td>
</tr>
<tr>
<td>( p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) )</td>
<td></td>
</tr>
<tr>
<td>( \neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q )</td>
<td></td>
</tr>
</tbody>
</table>
Show that $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

Left hand side

$\equiv \neg(p \lor (\neg p \land q))$

By De Morgan Law

$\equiv \neg p \land \neg(\neg p \land q))$

By De Morgan Law

$\equiv \neg p \land (\neg(\neg p) \lor \neg q))$

$\equiv \neg p \land (p \lor \neg q)$

$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$

$\equiv F \lor (\neg p \land \neg q)$

$\equiv (\neg p \land \neg q)$
Show that \((p \land q) \rightarrow (p \lor q)\) is a tautology.

Left hand side

\[
(p \land q) \rightarrow (p \lor q)
\]

since \(p \rightarrow q \equiv \neg p \lor q\)

\[
= \neg(p \land q) \lor (p \lor q)
\]

by De Morgan law

\[
= (\neg p \lor \neg q) \lor (p \lor q)
\]

by associative and commutative law

\[
= (\neg p \lor p) \lor (q \lor \neg q)
\]

\[
= T \lor T = T
\]
Propositional Logic, studied till now, cannot adequately express the meaning of statements in mathematics and in natural language.

- “Every computer is connected to the university network is functioning properly”
- “There is a computer on the university network that is under attack by an intruder”
- $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ for a positive integer $n$ (for all $n \geq 1$)

**Predicate logic** is the answer to such statements.
Statements involving variables are often found in mathematics assertions and in computer programs: \( x > 3; \ x = y + 3 \) and \( \text{int add (int x, int y)} \)

The statement “\( x \) is greater than 3 (\( x > 3 \))” has two parts: the variable \( x \) is the subject of the statement and the predicate “is greater than 3” refers to a property the subject can have.

We can denote the statement “\( x \) is greater than 3” by \( P(x) \) where \( P \) denotes the predicate “is greater than 3”. \( P(x) \) is also said to be the value of the propositional function \( P \) at \( x \).

Once a value has been assigned to \( x \), \( P(x) \) becomes a proposition and its truth value can be concluded.
Let $P(x)$ denotes the statement $x > 3$.
- Truth value of $P(4)$ is true
- Truth value of $P(2)$ is false

Let $Q(x, y)$ denotes the statement $x = y + 3$.
- Truth value of $Q(1, 2)$ is false as $1 = 2 + 3$ is false
- Truth value of $Q(3, 0)$ is true as $3 = 0 + 3$ is true

Let $R(x, y, z)$ denotes the statement $x + y = z$.
- Truth value of $R(1, 2, 3)$ is true as $1 + 2 = 3$ is true
- Truth value of $R(4, 6, 10)$ is also true.

What is the truth value of $P(5)$, $Q(0, -3)$ and $R(3, 4, 9)$
If \( x > 0 \) then \( x := x + 1 \)

- Let \( P(x) \) denotes \( x > 0 \). If \( P(x) \) is true for some value of \( x \), then the assignment operation is executed.
- In C program, we can write it as:

```c
int x;
if (x > 0)
    x = x + 1
```

A statement of the form \( P(x_1, x_2, x_3, \ldots, x_n) \) is the value of the propositional function \( P \) at the \( n \)-tuple \( (x_1, x_2, x_3, \ldots, x_n) \) and \( P \) is called a predicate.
QUANTIFIERS

- When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value.

- There is another way to create a proposition from a propositional function, called **Quantification**.

- Quantification expresses the extent to which a predicate is true over a range of elements.

- The area of logic that deals with predicates and quantification is called **Predicate Calculus**.
In English, the words all, some, many, none and few are used in quantification.

- “All boys in Lahore are student” can be translated into $\forall x: \text{a boy in Lahore } P(x)$ where $P$ is a predicate “is a student” (*Universal Quantifier*)
- “There is some student in Discrete class who will obtain an A” can translated into $\exists x: \text{a student of discrete class } Q(x)$ where $Q$ is a predicate “will obtain an A” (*Existential Quantifier*)

**Universal Quantification**: a predicate is true for every element under consideration

**Existential Quantification**: there are one or more elements for which the predicate is true.
Domain of discourse or Universe of discourse

“All boys in Lahore are student” can be translated into $\forall x$: a boy in Lahore $P(x)$ where $P$ is a predicate “is a student”: Domain of discourse consists of all boys of Lahore.

Universal Quantification

$P(x)$ for all values of $x$ in the domain. It is denoted by $\forall x$ $P(x)$ and is read as for all $x$ $P(x)$ or for every $x$ $P(x)$

An element $x$ for which $P(x)$ is false is called a counter example of $\forall x$ $P(x)$. If we can find at least one counter example, then the statement $\forall x$ $P(x)$ is false, otherwise it is true.
Universal Quantification

- Let $P(x)$ denotes the statement “$x + 1 > x$ for all real numbers”. It can be written as $\forall x \in \mathbb{R} \ x + 1 > x$. This statement is true for all values of $x$.

- Let $Q(x)$ denotes the statement $x < 2$ for all real numbers. It can be written as $\forall x \in \mathbb{R} \ x < 2$. This statement is false as $Q(3)$ is false (counter example).

- Let $P(x)$ denotes $x^2 \geq 1$. $\forall x \ P(x)$ for $x$ belong to set $\{1, 2, 3, 4\}$ (domain of discourse)
  - $P(1) \land P(2) \land P(3) \land P(4) = True$
  - It means $\forall x \ P(x)$ is true for the said domain
Universal Quantification

- For all positive numbers $x$, $1 + 2 + 3 + \ldots + x = \frac{x(x+1)}{2}$ (proof by Mathematical Induction) domain of discourse consists of positive integers
Existential Quantifier

- “There exist an element x in the domain such that P(x)”.
  It denoted by $\exists x \ P(x)$ and is read as there is an x such that P(x) or there is at least one x such that P(x).

- Let P(x) denotes $x > 3$. $\exists x \ P(x)$, where domain consists of all real numbers.
  - P(x) is true for all real numbers greater than 3, but it is false for real numbers less than or equal to 3.

- Let Q(x) denotes $x = x + 1$. $\exists x \ Q(x)$ is false over the domain to real numbers.
Existential Quantification

- Let $P(x)$ denotes $x^2 > 10$. $\exists x\ P(x)$ for the domain $\{1, 2, 3, 4\}$
  - $P(1) \lor P(2) \lor P(3) \lor P(4) = F \lor F \lor F \lor T = True$
  - It means $\exists x\ P(x)$ is true as there exists one value for which $P(x)$ is true
The quantifiers $\forall$ and $\exists$ have higher precedence than all other logical operators.

- $\forall x \, P(x) \land Q(x)$ means Conjunction of ($\forall x \, P(x)$) and $Q(x)$
- In other words, it means ($\forall x \, P(x)$) $\land$ $Q(x)$ rather than $\forall x \, (P(x) \land Q(x))$
When a quantifier is used on a variable, then the variable is **bound**.

An occurrence of a variable that is not bound by a quantifier, is called **free**.

Example

\( \forall x \ P(x, y) \) here \( x \) is a bounded variable while \( y \) is a free variable

The part of a logical expression to which a quantifier is applied is called **the scope of the quantifier**.

Example

\( \forall x \ (P(x) \land Q(x)) \lor R(x) \)

Here Scope of Universal Quantifier (\( \forall \)) is on \( P(x) \land Q(x) \) and not on \( R(x) \)
LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into those statements and which domain of discourse is used for the variables in those propositional functions.

- Let S and T are two statements involving predicates and quantifiers. S and T are logically equivalent if and only if they have same truth values and it is denoted by $S \equiv T$. 
Examples:

- $\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$
- $\exists x (P(x) \land Q(x)) \equiv \exists x P(x) \land \exists x Q(x)$

Negation

- “Every student in this class has taken a course of Calculus”
- What is the negation of this statement?
- “No student of this class has taken a course of Calculus”
**Logical Equivalences Involving Quantifiers**

**Negation**

- Let $P$ is a predicate that “has taken a course of Calculus”
- $x$ represent a student of this class.
- The statement “Every student in this class has taken a course of Calculus” can be written as $\forall x \ P(x)$
- $\neg (\forall x \ P(x)) \equiv \text{“No student of this class has taken a course of Calculus”}$
- $\exists x \neg P(x) \equiv \text{“There exist no student of this class who has taken a course of Calculus”}$

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**LOGICAL EQUIVALENCES INVOLVING QUANTIFIERS**

**Negation**

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**TABLE 2  De Morgan’s Laws for Quantifiers.**

<table>
<thead>
<tr>
<th>Negation</th>
<th>Equivalent Statement</th>
<th>When Is Negation True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg \exists x \ P(x)$</td>
<td>$\forall x \neg P(x)$</td>
<td>For every $x$, $P(x)$ is false.</td>
<td>There is an $x$ for which $P(x)$ is true.</td>
</tr>
<tr>
<td>$\neg \forall x \ P(x)$</td>
<td>$\exists x \neg P(x)$</td>
<td>There is an $x$ for which $P(x)$ is false.</td>
<td>$P(x)$ is true for every $x$.</td>
</tr>
</tbody>
</table>
“Every student in this class has taken a course of Calculus”

- We can rewrite this statement as “For every student in this class, that student has studied Calculus”

- Now we introduce a variable so that statement becomes: “For every student \( x \) in this class, \( x \) has studied Calculus”
“For every student $x$ in this class, $x$ has studied Calculus”

**One way:**

- Let $P$ is a predicate that “$x$ has taken a course of Calculus”
- $x$ represent a student of this class, **domain of discourse** is specified (**Restricted Domain**).
- Logical Expression: $\forall x \ P(x)$
“For every student x in this class, x has studied Calculus”

Second way:

- S(x) is propositional function that x is a student of this class.
- C(x) is propositional function that x has studied Calculus.
- x is a human, specification of domain of discourse
- The statement can be expressed as

\[ \forall x (S(x) \rightarrow C(x)) \]

It cannot be written as \[ \forall x (S(x) \land C(x)) \]
Exercise

“Some student of this class has visited Europe.”

Solutions:

1. $\exists x \ P(x, \text{Europe})$ where $P(x, y)$ means “$x$ has visited $y$” and $x$ represents the set of students of this class (restricted domain of discourse)

2. $\exists x \ (S(x) \rightarrow V(x, \text{Europe}))$ where $S(x)$ means “$x$ is student of this class”, $V(x,y)$ means “$x$ has visited $y$” and $x$ represents all human beings (domain of discourse)

It can be written as $\exists x \ (S(x) \land V(x, \text{Europe}))$
Exercise

“Every student of this class has visited Gulberg.”

Solutions:

1. $\forall x \ P(x, \text{Gulberg})$ where $P(x, y)$ means “x has visited y” and x represents the set of students of this class (restricted domain of discourse)

2. $\forall x \ (S(x) \rightarrow V(x, \text{Gulberg}))$ where $S(x)$ means “x is student of this class”, $V(x, y)$ means “x has visited y” and x represents all human beings (domain of discourse)

It cannot be written as

$\forall x \ (S(x) \land V(x, \text{Europe}))$
Two quantifiers are nested, if one is written within the scope of the other.

- $\forall x \exists y \ (x + y = 0)$
- $\forall x \forall y \ (x + y = y + x) \quad \text{– Commutative Law}$

**Order of Quantification**

The order of quantifiers is important unless all quantifiers are universal or existential.

- $\forall x \forall y \ (x + y = y + x) \equiv \forall y \forall x \ (x + y = y + x)$
- $\forall x \exists y \ (x + y = 0)$  
  not equivalent  $\exists y \forall x \ (x + y = 0)$
- $\forall x \exists y \ (x + y = 0)$ means “For all real No. $x$, there is a real No. $y$ such that $x + y = 0$” and this is true.
- $\exists y \forall x \ (x + y = 0)$ means “There is a real No. $y$ such that for all real No. $x$, $x + y = 0$” and this is false.
TABLE 1 Quantifications of Two Variables.

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \forall y P(x, y)$</td>
<td>$P(x, y)$ is true for every pair $x, y$.</td>
<td>There is a pair $x, y$ for which $P(x, y)$ is false.</td>
</tr>
<tr>
<td>$\forall y \forall x P(x, y)$</td>
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<td>There is a pair $x, y$ for which $P(x, y)$ is false.</td>
</tr>
<tr>
<td>$\forall x \exists y P(x, y)$</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is true.</td>
<td>There is an $x$ such that $P(x, y)$ is false for every $y$.</td>
</tr>
<tr>
<td>$\exists x \forall y P(x, y)$</td>
<td>There is an $x$ for which $P(x, y)$ is true for every $y$.</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is false.</td>
</tr>
<tr>
<td>$\exists x \exists y P(x, y)$</td>
<td>There is a pair $x, y$ for which $P(x, y)$ is true.</td>
<td>$P(x, y)$ is false for every pair $x, y$.</td>
</tr>
<tr>
<td>$\exists y \exists x P(x, y)$</td>
<td>$P(x, y)$ is false for every pair $x, y$.</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is false.</td>
</tr>
</tbody>
</table>